

4 Equilibrium

 Name

Learning objectives

- List the main assumptions of game theory
- Define and formulate strategic games
- Define Nash equilibrium, explain its applications and find the solution
- Define basic microeconomics concepts such as producer and consumer surplus, market equilibrium and social welfare
- Explain the difference between centralized operation and competition in power systems
- Explain the concepts of perfect and imperfect competition. Formulate problems that model these two situations and solve them
- Define, formulate and solve models with Cournot players

Bibliography

- Basic reading:
 - Osborne, Martin J., and Ariel Rubinstein. A course in game theory. MIT press, 1994 (chapters 1 and 2).
 - Gabriel, S., Conejo. A., Fuller, D., Hobbs, B., and Ruiz, C. Complementarity modeling in energy markets. Springer, 2012 (chapter 3).
 - Kirschen, D. S., and G. Strbac. Fundamentals of power system economics, 2004 (chapters 1 and 2).
 - Zhang, X. P. Restructured Electric Power Systems: Analysis of Electricity Markets with Equilibrium Models, 2010 (chapter 2).
- Further reading:
 - Neumann, Ludwig Johann, and Oskar Morgenstern. Theory of games and economic behavior. Vol. 60. Princeton, NJ: Princeton university press, 1947.
 - Nash, John F. Equilibrium points in n-person games. Proceedings of the national academy of sciences 36, no. 1 (1950): 48-49.

Game theory

- Definition: Game theory is the set of analytical tools designed to help us understand the phenomena that we observe when decision-makers interact.
- In game theory analysis decision-makers are assumed to be:
 - Rational, which means that they have a well-defined objective function
 - Strategic, which means that they take into account the actions of other decision-makers
- Games can be classified into
 - Cooperative/Noncooperative. In cooperative games players can make coordinated actions to maximize their objective functions. In noncooperative games the set of actions of each player is primitive.
 - Simultaneous/Sequential. In simultaneous games (or strategic games) each player chooses his plan of action once and for all, and all players' decisions are made simultaneously. In sequential games (or extensive-form games) players determine their actions sequentially and later players have some knowledge about earlier actions of other players.
 - Perfect/Imperfect information. Within sequential games we can have games with perfect information, if all players know the moves previously made by all other players, or with imperfect information, if information about previous actions is not perfectly known.
- A rational model includes the following elements for each player:
 - A set $\mathcal{A}$ of actions
 - A set $\mathcal{C}$ of consequences
 - A consequence function $g : \mathcal{A} \rightarrow \mathcal{C}$
 - A preference relation $\succsim$ on $\mathcal{C}$
 - A utility function $u : \mathcal{C} \rightarrow \mathbb{R}$ such that $c_1 \succsim c_2 \Leftrightarrow u(c_1) \geq u(c_2)$
- Each player makes its decisions by solving the following optimization problem

$$\begin{array}{ll} \underset{a}{\text{maximize}} & u(g(a)) \\ \text{subject to} & a \in \mathcal{A} \end{array}$$

Example 4-1: Prisoner's dilemma (I)

Two members of a criminal gang are arrested and imprisoned. Each prisoner is in solitary confinement with no means of communicating with the other. The prosecutors lack sufficient evidence to convict the pair on the principal charge. They hope to get both sentenced to a year in prison on a lesser charge. The prosecutors offer each prisoner a bargain. Each prisoner is given the opportunity either to: betray the other by testifying that the other committed the crime, or to cooperate with the other by remaining silent. Classify this game.

Strategic games

A strategic game is a model in which each decision-maker chooses his plan of action once and for all and these choices are made simultaneously by all players. The main elements of this game are:

- A set of available actions $\mathcal{A}_i$ for each player
- A preference relation $\succsim_i$ on $\mathcal{A} = \times_{j \in N} \mathcal{A}_j$
- Sometimes an utility function $u_i : \mathcal{A} \rightarrow \mathbb{R}$ such that $u_i(a) \geq u_i(b) \Leftrightarrow a \succsim_i b$

A finite strategic game with two players can be described conveniently in a table as follows

		$\mathcal{A}_1$	
		L	R
$\mathcal{A}_2$	T	u_1, u_2	v_1, v_2
	B	w_1, w_2	y_1, y_2

Nash equilibrium

The Nash equilibrium of a strategic game is the steady state in which none of the players has an incentive to change its decisions. Mathematically, the set of actions $a^* \in \mathcal{A}$ is a Nash equilibrium if

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*) \quad \forall a_i \in \mathcal{A}_i, \quad \forall i$$

Existence of Nash equilibrium in pure strategies

A strategic games with N players, set of actions $\mathcal{A}_i$ for each player, and utility functions $u_i : \mathcal{A}_i \rightarrow \mathbb{R}$ has a Nash equilibrium in pure strategies if

- each set of actions $\mathcal{A}_i$ is a nonempty, closed, bounded and convex set
- the utility function u_i are continuous and concave or quasi-concave

Example 4-2: Prisoner's dilemma (II)

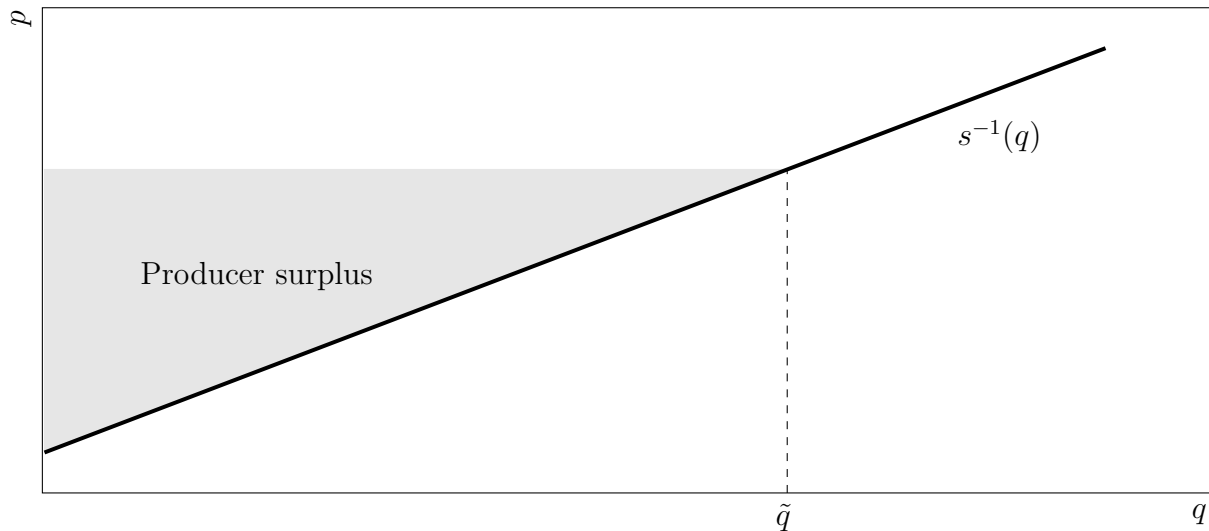
The offer of the prosecutors to each member of the criminal gang is the following. If A and B each betray the other, each of them serves 2 years in prison. If A betrays B but B remains silent, A will be set free and B will serve 3 years in prison. If A and B both remain silent, both of them will only serve 1 year in prison. Fill in the table below and find the Nash equilibrium.

		A	
B		/	/
		/	/

Supply function and producer surplus

A supply curve provides the quantity q of a good that a firm is willing to produce as a function of the price p of that good $q = s(p)$. The inverse supply curve provides the required price p to incentive a given production q , i.e., $p = s^{-1}(q)$. Under perfect competition, the inverse supply function coincides with the marginal cost function. The producer surplus $\mathcal{PS}$ for a quantity $\tilde{q}$ is computed as

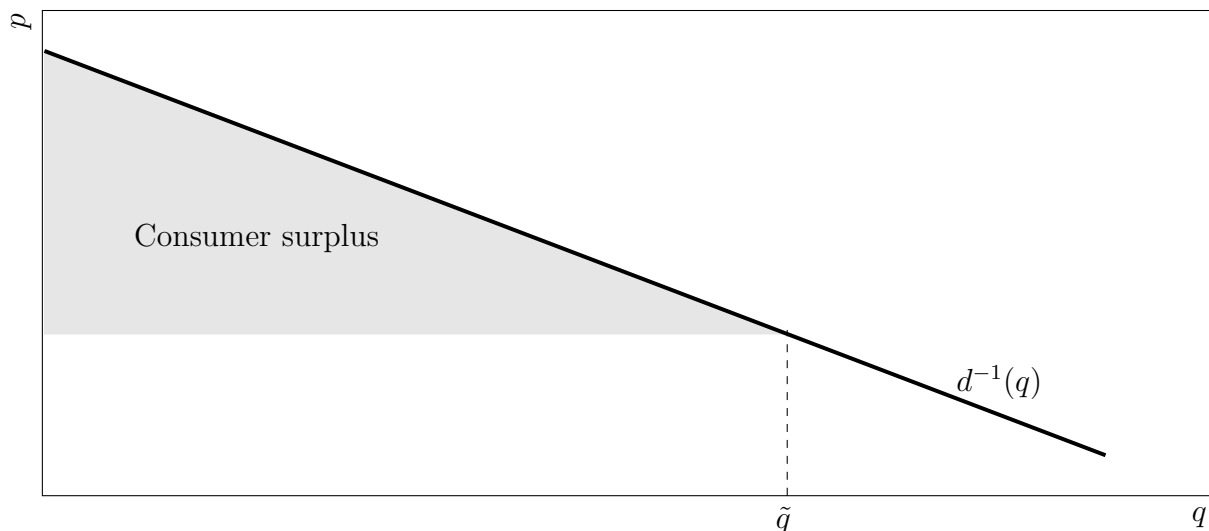
$$\mathcal{PS} = p\tilde{q} - \int_0^{\tilde{q}} s^{-1}(q) dq$$



Demand function and consumer surplus

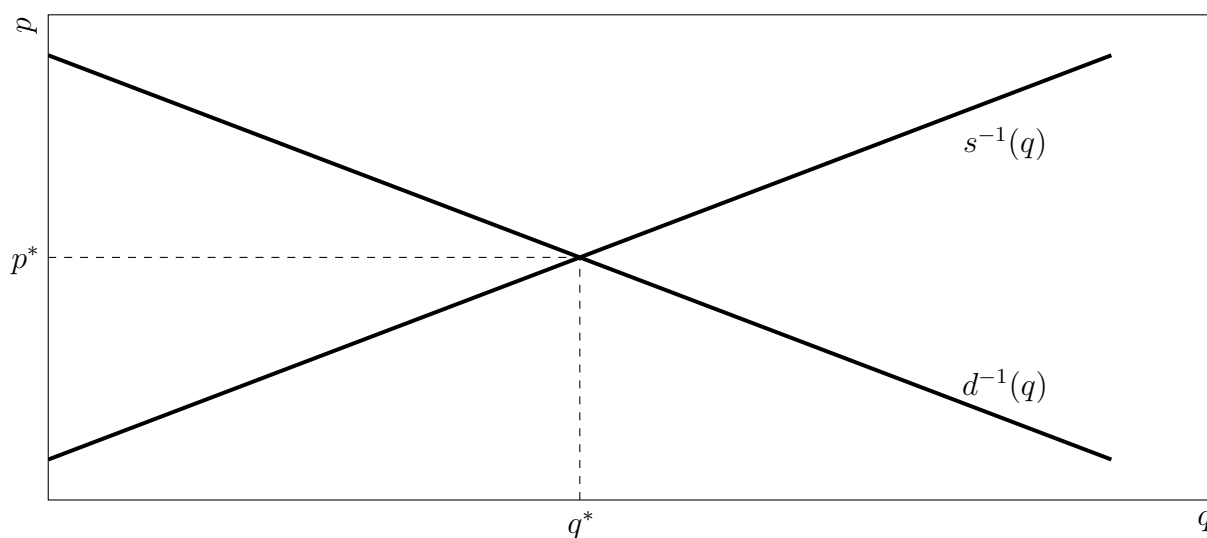
The demand function relates the demand of a good q as a function of the price p , i.e., $q = d(p)$. The inverse demand function (depicted below) provides the price level p required to create a demand d , i.e., $p = d^{-1}(q)$. Under perfect competition, the inverse demand function coincides with the marginal utility of the consumers. The consumer surplus $\mathcal{CS}$ for a quantity $\tilde{q}$ is computed as

$$\mathcal{CS} = \int_0^{\tilde{q}} d^{-1}(q) dq - p\tilde{q}$$



Market equilibrium

The equilibrium market price is such that the quantity that the suppliers are willing to produce is equal to the quantity that the consumers are willing to buy. The figure below plots the equilibrium price p^* and quantity q^* .

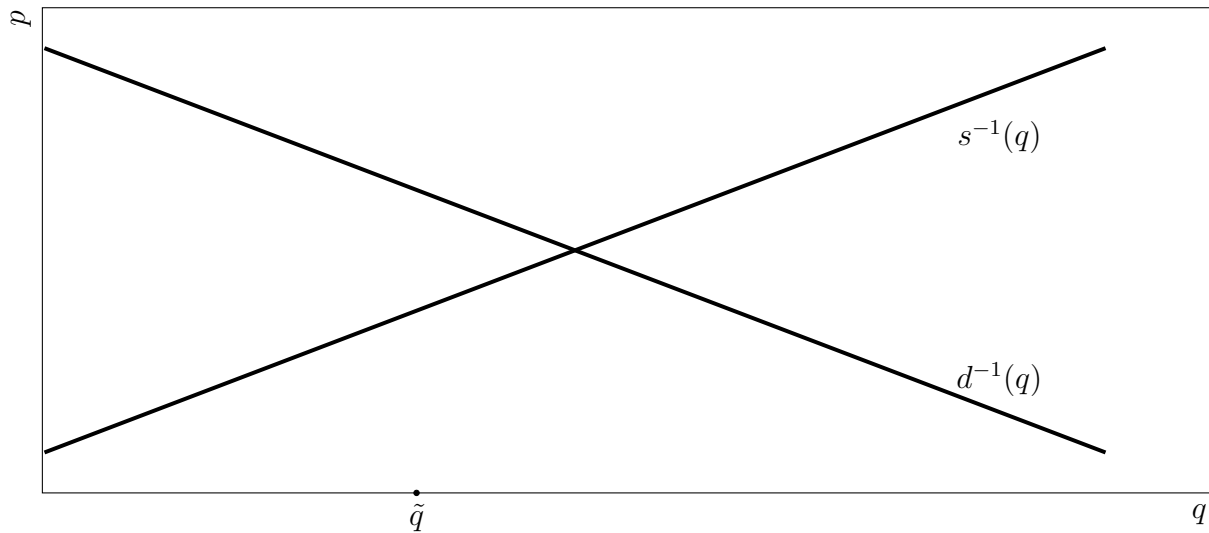


Example 4-3: Market equilibrium

We consider a linear inverse demand function $p = \alpha - \beta q$ and a linear inverse supply function $p = \gamma + \delta q$, with $\alpha, \beta, \gamma, \delta \geq 0$. Determine, for $\alpha = 50$, $\beta = 0.2$, $\gamma = 15$, $\delta = 0.5$, the equilibrium price and quantity as well as the producer and consumer surplus.

Social welfare

The social welfare $\mathcal{SW}$ is defined as the sum of the consumer surplus plus the producer surplus. If the amount produced and consumed is equal to $\tilde{q}$, draw in the figure below the social welfare.



Example 4-4: Equilibrium as maximization of social welfare

We consider a linear inverse demand function $p = \alpha - \beta q$ and a linear supply function $p = \gamma + \delta q$, with $\alpha, \beta, \gamma, \delta \geq 0$. Formulate an optimization problem that maximizes the social welfare with q as the decision variable. Express the optimality condition of such optimization problem and solve it for $\alpha = 50$, $\beta = 0.2$, $\gamma = 15$, $\delta = 0.5$.

📌 Example 4-5: Social welfare maximization

In Example 1-14 an economic dispatch problem was formulated to minimize the production cost of satisfying a constant demand d . In this example, instead of a fixed demand we consider a demand that changes with the price according to an inverse demand function $p = f_d^{-1}(d)$ where p and d represent the electricity price and the demand level, respectively. The utility of the demand is therefore computed as

$$\mathcal{U} = \int_0^d f_d^{-1}(d) dd$$

We also have N^G generating units with a production cost function $\mathcal{C}_g(x_g)$ and capacity $\bar{x}_g$. Use $\underline{\eta}_g, \bar{\eta}_g$ as the dual variables corresponding to the production limits of the unit g , and λ as the dual variable corresponding to the balance equation. Formulate the optimization problem that maximizes the social welfare and provide its KKT conditions.

Centralized electricity sector

- Based on vertically integrated utilities that managed the generation, transmission, and distribution of electricity and were operated as monopolies.
- The vertically integrated utility needed to ensure that there was adequate power supply to meet the demand load while satisfying the technical limits of its assets such as the capacities of generators and transmission lines.
- As load demands increased, vertically integrated utilities upgraded existing facilities or constructed new transmission corridors to maintain the reliable delivery of electricity.

Example 4-6: Mathematical tools for centralized electricity sector

Among the mathematical models presented in this course, which one fits better with the centralized operation of power systems?

Restructured electricity sector

- The main purpose of restructuring is to let economic forces drive the price of electric supply and maximize social welfare via competition.
- In a restructured or deregulated electricity market, a traditional utility is separated into three generic entities: generator companies (GENCOs), transmission companies (TRANSCOs), and the distribution companies (DISCOs).
- GENCOs operate and maintain existing generating plants and sell their production through bilateral contracts or organized markets.
- A TRANSCO is an entity responsible for building, maintaining, and operating the transmission systems in a certain geographical region.
- DISCOs receive bulk energy from the transmission grid and distribute the electricity through their facilities to customers in a certain geographical region.
- The Independent System Operator (ISO) is a neutral, independent entity responsible for maintaining secure and economic operation of an open access transmission system on a regional basis. The ISO is thus responsible for maintaining the energy balance of the system.
- Most electricity market designs include both bilateral contracts and centralized auctions.
- Producers and consumers can participate in centralized auctions by submitting offers and bids to sell and buy electricity, respectively. The market is cleared to maximize the social welfare.
- Each market player must determine the strategy that maximizes its profit.

Example 4-7: Mathematical tools for restructured electricity sector

Among the mathematical models presented in this course, which one fits better with the competitive operation of power systems?

Market power and competition

- Market power is the ability of a firm to profitably raise the market price of a good.
- A price-taker producer cannot exercise market power.
- If all producers are price-takers, we have perfect competition.
- A price-maker producer can exercise market power.
- If some producers are price-makers, we have imperfect competition. Forms of imperfect competition include:
 - Monopoly: only one price-maker supplier
 - Duopoly: two price-maker suppliers
 - Oligopoly: few price-maker suppliers

Example 4-8: Price-taker producer

Formulate an optimization problem to maximize the profit of a price-taker power producer g with a production cost function $\mathcal{C}_g(x_g)$ and capacity $\bar{x}_g$. Electricity price is denoted by p . (Similar to Example 1-13). Then formulate its KKT conditions. How would the KKT change if we consider a price-maker producer?

Example 4-9: Equilibrium of price-taker producers

Formulate the equilibrium model of N^G price-takers producers. Assume that the electricity price p is given by the inverse demand function is given by $f_d^{-1}(d)$. Which kind of problem did you obtain? Compare this formulation with the one in Example 4-5 and discuss the implications.

📌 Example 4-10: Monopoly model

Formulate a profit maximization model (similar to Example 4-8) assuming that all producers merge into one single firm that anticipates the effect of the total production on the market price, which is formulated as the inverse demand function $f_d^{-1}(d)$. Use $\mathcal{C}_g(x_g)$ as the production cost function of the units, $\underline{\eta}_g, \bar{\eta}_g$ as the dual variables corresponding to the production limits of the units, and λ as the dual variable corresponding to the balance equation. Determine the KKT of that optimization problem. Can you use λ as the price?

Cournot competition

Cournot competition is an economic model used to describe an industry structure with the following features:

- Different firms produce a homogeneous product
- Firms compete in quantities to maximize profit
- Firms determine produced quantities simultaneously
- Firms can exercise market power and affect the resulting price through production decisions
- Each firm assumes that its own output decision do not affect the decisions of its rivals

Example 4-11: Nash-Cournot model

Formulate a profit maximization model of one Cournot power producer with a convex cost function $C_g(x_g)$. The inverse demand function is given by $f_d^{-1}(x_g + x_{-g})$, where x_{-g} denotes the sum of the amounts produced by the rest of the players. Then, determine the corresponding KKT conditions taking x_{-g} as given.

 Solving Example 4-11 with GAMS (Cournot.gms)

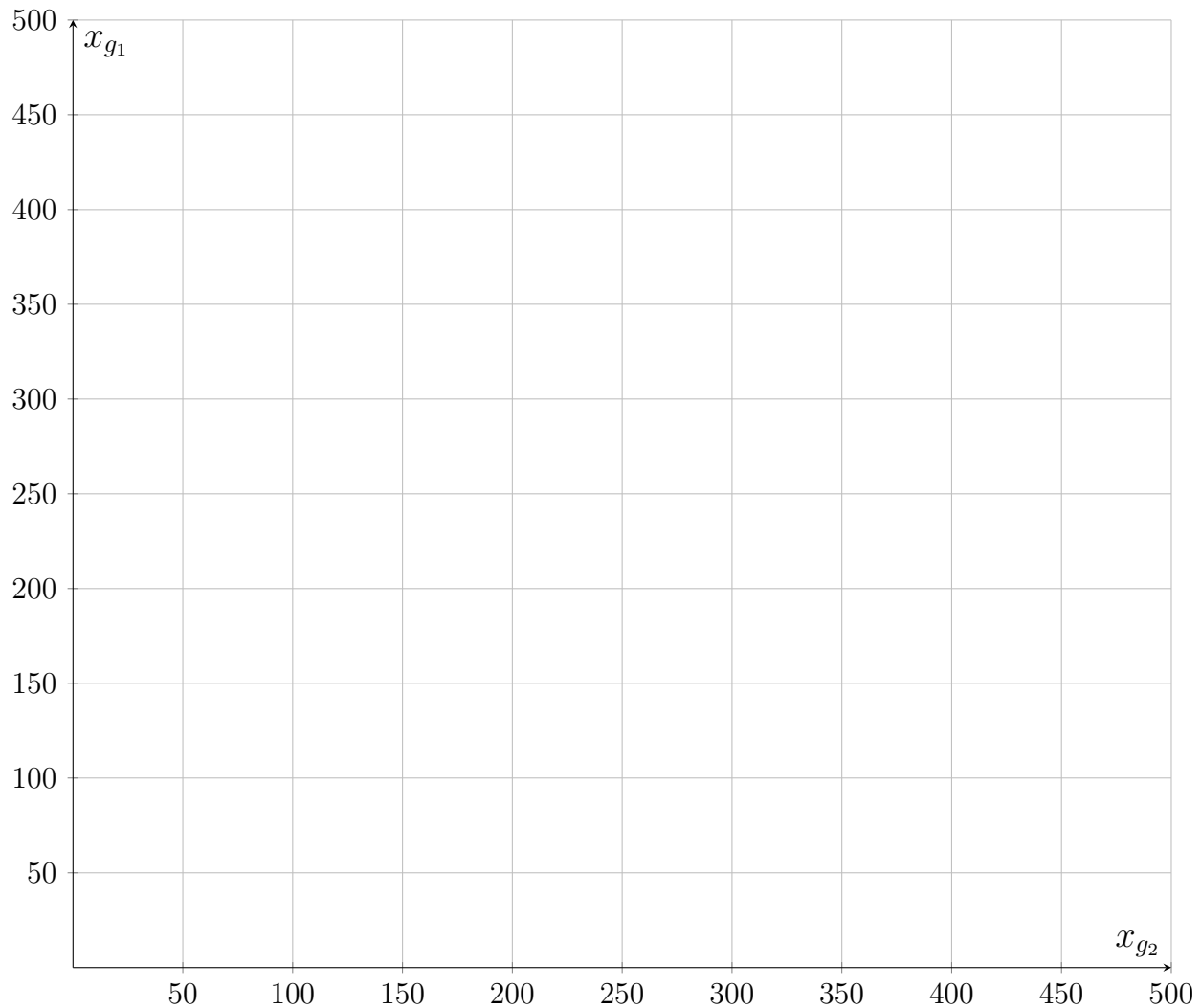
Let us assume two power producers g_1 and g_2 with quadratic cost functions $\mathcal{C}_g(x_g) = a_g x_g^2 + b_g x_g + c_g$ and a linear inverse demand function $f_d^{-1}(x_g + x_{-g}) = \alpha - \beta(x_g + x_{-g})$. Use GAMS to solve the optimization problem corresponding to Example 4-11 for g_1 and complete the table below. Use the following data $a_{g1} = 0.01$, $b_{g1} = 5$, $c_{g1} = 100$, $\bar{x}_{g1} = 500$, $\alpha = 70$, $\beta = 0.1$.

x_{g2}	0	100	200	300	400	500
x_{g1}						

Use the same code and fill in the table below for a g_2 with the following data: $a_{g2} = 0.02$, $b_{g2} = 10$, $c_{g2} = 100$, $\bar{x}_{g2} = 500$, $\alpha = 70$, $\beta = 0.1$.

x_{g1}	0	100	200	300	400	500
x_{g2}						

Then sketch below the optimal quantity produced by g_1 as a function of the production of g_2 , and the optimal quantity produced by g_2 as a function of the production of g_1 and find the equilibrium.



Example 4-12: Nash-Cournot equilibrium

Formulate a complementarity model to determine the equilibrium of several power Cournot power producers using their corresponding KKT conditions. Compare with the perfect-competition formulation obtained in Example 4-9.

Solving Example 4-12 with GAMS (CournotEqui.gms)

Use GAMS to solve the complementarity problem of Example 4-12 for the data corresponding to g_1 and g_2 provided in Example 4-11. Is the equilibrium close to the one you inferred graphically?

Comparing Nash-Cournot model

For this exercise you need to code in GAMS the models corresponding to Example 4-10 (Monopoly.gms) and Example 4-9 (PerfectCom.gms) and also use the code CournotEqui.gms to compare the perfect competition and Nash-Cournot models. You have to run both models using the following data: $\alpha = 70$, $\beta = 0.1$,

Unit	$\underline{x}_g$	$\bar{x}_g$	a_g	b_g	c_g
g_1	0	250	0.01	5	100
g_2	0	150	0.02	10	100
g_3	0	100	0.03	16	100

and fill in the following table with the results:

	Perfect competition (PerfectCom.gms)	Nash-Cournot (CournotEqui.gms)	Monopoly (Monopoly.gms)
x_{g1}			
x_{g2}			
x_{g3}			
d			
price			
social welfare			
producer surplus			
consumer surplus			

Comment the results above.

✚ Example 4-13: Generation expansion of power producers

Until now, the maximum capacity of the power producers denoted by $\bar{x}_g$ was a given parameter. However, in a decentralized environment power producers must also determine their optimal generation capacity to maximize their profits. This problem is known as generation expansion problem. Generation expansion problems usually cover a long planning horizon (one to several years) that is divided into shorter time periods (of one hour, for example), normally denoted with the index t . In this way, production decisions x_{gt} depend now on the time period t since they can change from hour to hour; while investment decisions $\bar{x}_g$ do not depend on t because it is assumed that the capacity will be the same for the whole planning horizon. Assuming a production cost function given by $\mathcal{C}_g^P(x_{gt})$, an investment cost function given by $\mathcal{C}_g^I(\bar{x}_g)$ and a linear inverse demand function given by $\alpha_t - \beta_t(x_{gt} + x_{-gt})$, formulate an optimization problem to obtain the optimal capacity and production levels of a power producer. Determine its corresponding KKT conditions and discuss how they would change depending on whether the power producer is price-taker or behaves a la Cournot. *Tip: assume that the minimum production is equal to 0.*

📌 Example 4-14: Generation expansion equilibrium of power producers

Using the same assumptions as in 4-13, formulate the equilibrium problem to determine the generation expansion decisions of a set of power producers.

📁 Solving Example 4-14 with GAMS (GenExpEqu.gms)

Use GAMS to solve the complementarity problem of Example 4-14 for two power producers with the following production and investment cost functions

$$\mathcal{C}_{g_1}^I(\bar{x}_{g_1}) = 100\bar{x}_{g_1} \quad \mathcal{C}_{g_1}^P(x_{g_1t}) = 10x_{g_1t} \quad \mathcal{C}_{g_2}^I(\bar{x}_{g_2}) = 50\bar{x}_{g_2} \quad \mathcal{C}_{g_2}^P(x_{g_2t}) = 20x_{g_2t}$$

The value of β_t is equal to 1 for the ten time periods while the value of α_t varies throughout time as indicated below

t	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
α_t	10	20	30	40	50	60	70	80	90	100

Fill in the following table and comment the results.

	$\bar{x}_{g_1}$	$\bar{x}_{g_2}$	profit g_1	profit g_2	social welfare
Price-takers					
Cournot					